Inequality 3

1. Prove that $(a + b)(b + c)(c + a) \geq 8abc$ is true for all positive numbers a, b, and c with equality only if $a = b = c$.

Let a, b, c be positive numbers.

$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0 \iff a + b \geq 2\sqrt{ab}$ with equality holds only if $a = b$

Similarly, $b + c \geq 2\sqrt{bc}$ with equality holds only if $b = c$

$c + a \geq 2\sqrt{ca}$ with equality holds only if $c = a$

Hence $(a + b)(b + c)(c + a) \geq \left(2\sqrt{ab}\right)\left(2\sqrt{bc}\right)\left(2\sqrt{ca}\right) = 8abc$

with equality holds only if $a = b = c$.

2. Solve $\frac{x(x+2)}{x-1} \leq 0$.

$$\frac{x(x+2)}{x-1} \leq 0 \Longrightarrow \begin{cases} x(x+2) \geq 0 \\ x-1 < 0 \end{cases} or \begin{cases} x(x+2) \leq 0 \\ x-1 > 0 \end{cases}$$

$$\Longrightarrow \begin{cases} -2 \geq x \; or \; x \geq 0 \\ x < 1 \end{cases} or \begin{cases} -2 \leq x \leq 0 \\ x > 1 \end{cases}$$

$$\Longrightarrow (-2 \geq x \; or \; 0 \leq x < 1) \; or \; no \; solution$$

$$\Longrightarrow \boldsymbol{-2 \geq x \; or \; 0 \leq x < 1}$$

3. Solve $\frac{9}{1-x} \leq \frac{7x+5}{x+3}$.

$$\frac{7x+5}{x+3} + \frac{9}{x-1} \geq 0$$

$$\frac{7x^2+7x+22}{(x+3)(x-1)} \geq 0 \ldots (*)$$

Method 1

Consider: $f(x) = 7x^2 + 7x + 22$. Since Δ of $f(x) = 7^2 - 4(7)(22) = -576 < 0$

The curve $f(x)$ cannot cut the axis and is above the x-axis since $a = 7 > 0$.

$\therefore 7x^2 + 7x + 22 > 0$ for all $x \in \boldsymbol{R}$.

Method 2

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a} = 7\left(x + \frac{7}{14}\right)^2 - \frac{-576}{28} = 7\left(x + \frac{1}{2}\right)^2 + \frac{144}{7} > 0 + \frac{144}{7} > 0$$

Hence from $(*)$, $(x+3)(x-1) > 0 \Longrightarrow \boldsymbol{x} < -3 \; or \; x > 1$.

4. Solve $|2x - 3| < |x - 1| + |x - 2|$ for x.

$|2x - 3| < |x - 1| + |x - 2| \Longrightarrow |(x-1) + (x-2)| < |x - 1| + |x - 2|$

Since both sides are positive, we square both sides,

$(x-1)^2+2(x-1)(x-2)+(x-2)^2<(x-1)^2+2|(x-1)(x-2)|+(x-2)^2$
$(x-1)(x-2)<|(x-1)(x-2)|$
This is a strict inequality, we only have $(x-1)(x-2)<0 \Rightarrow 1<x<2$

5. (a) Find the solution of the general symmetric inequality:

$$|x+a|+|x-a|\le b,\ \ b\ge 0$$

(b) Hence find x where $|x-1|+|x-2|\le 4$.

(a) $|x+a|+|x-a|\le b,\ \ b\ge 0 \ldots (1)$

By triangular inequalities $|x+a|+|x-a|\ge |(x+a)+(x-a)|=2|x|$

(1) becomes: $|x|\le \frac{b}{2} \Leftrightarrow -\frac{b}{2}\le x\le \frac{b}{2}$

Note also, $|x+a|+|x-a|=|x+a|+|a-x|\ge |(x+a)+(a-x)|=2|a|$

(1) becomes: $|a|\le \frac{b}{2} \Leftrightarrow -\frac{b}{2}\le a\le \frac{b}{2}$

In general, (1) has solution: $\boldsymbol{-\frac{b}{2}\le x\le \frac{b}{2}}$, if $-\frac{b}{2}\le a\le \frac{b}{2}$.

(b) Now we go back to $|x-1|+|x-2|\le 4 \ldots (2)$

(2) obviously is not the same as the symmetric form as in (1).

Put $u=x-1.5$, we get $|u+0.5|+|u-0.5|\le 4$ (-1.5 is the mid point of -1 and -2)

Apply previous result, $-2\le u\le 2 \Rightarrow -2\le x-1.5\le 2 \Rightarrow \boldsymbol{-0.5\le x\le 3.5}$

6. Solve $\left|x-\frac{1}{x}\right|<4$.

We assume x is real.

Since both sides are positive, squaring gives $\left(x-\frac{1}{x}\right)^2<16$

Since $x\ne 0$, multiple both sides by $x^2>0$, $(x^2-1)^2<16x^2$

Hence $(x^2-1)^2-(4x)^2<0$

$\Rightarrow (x^2-4x-1)(x^2+4x-1)<0$

$\Rightarrow \begin{cases} x^2-4x-1>0 \\ x^2+4x-1<0 \end{cases}$ or $\begin{cases} x^2-4x-1<0 \\ x^2+4x-1>0 \end{cases}$

$\Rightarrow \begin{cases} [x-(2-\sqrt{5})][x-(2+\sqrt{5})]>0 \\ [x-(-2-\sqrt{5})][x-(-2+\sqrt{5})]<0 \end{cases}$ or $\begin{cases} [x-(2-\sqrt{5})][x-(2+\sqrt{5})]<0 \\ [x-(-2-\sqrt{5})][x-(-2+\sqrt{5})]>0 \end{cases}$

$\Rightarrow \begin{cases} 2-\sqrt{5}>x \text{ or } x>2+\sqrt{5} \\ -2-\sqrt{5}<x<-2+\sqrt{5} \end{cases}$ or $\begin{cases} 2-\sqrt{5}<x<2+\sqrt{5} \\ -2-\sqrt{5}>x \text{ or } x>\sqrt{5}-2 \end{cases}$

$\Rightarrow \boldsymbol{-2-\sqrt{5}<x<2-\sqrt{5}}$ ***or*** $\boldsymbol{\sqrt{5}-2<x<2+\sqrt{5}}$

7. Prove by mathematical induction the inequality $5^n \geq n^5$ for all $n \geq 5$.

Let $P(n): 5^n \geq n^5$

For (5) , $5^5 \geq 5^5$ is obviously true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}, k \geq 5$, that is $5^k \geq k^5 \ldots (1)$

For $P(k+1)$,

$$5^{k+1} = 5 \cdot 5^k \geq 5k^5 \quad \text{, by (1)}$$
$$= k^5 + k^5 + k^5 + k^5 + k^5$$
$$= k^5 + kk^4 + k^2k^3 + k^3k^2 + k^4k \quad , k \geq 5$$
$$\geq k^5 + 5k^4 + 25k^3 + 125k^2 + 624k + k \quad , \; k \geq 5$$
$$\geq k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 \quad , k \geq 5$$
$$= (k+1)^5$$

$\therefore P(k+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}, n \geq 5$.

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